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ON THE PROBLEM OF TERMINATING A GAME AT THE FIRST INSTANCE OF ABSORPTION*

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The sufficient conditions are established for terminating a game at the first instant of absorption /1/. The satisfaction of these conditions ensures the termination of the integration procedure /2/, using the operator of programmed absorption at the first step. Examples are given.

1. The concept of the first instant of absorption was introduced in /1/. The possibility of terminating a game in a number of cases in a time equal to the duration of the first instant of absorption was shown in /3/. With the introduction of the operator T of programmed absorption /4-6/, the possibility of terminating a game with the terminal set X in a time equal to the duration of the first instant of absorption follows from the inclusion

$$T_t^\tau (T_\tau^b (X)) \supset T_t^b (X), \quad t < \tau < b \tag{1.1}$$

The form of operator T depends on the class of admissible programmed controls of the players /5/.

Consider a game formulated as follows. The game phase space Z, a segment $[a, b]$ of the numerical axis, and the sets P and Q of controls are specified. With each position $z \in Z$ and each pair of numbers $a \leqslant t < \tau \leqslant b$ we place in correspondence the set $V(z, t, \tau)$ of programmed controls $v: [t, \tau] \to Q$ of the second player. With each programmed control $v(\cdot)$ of that set we place in correspondence some set $U(z, t, \tau, v(\cdot))$ of programmed controls $u: [t, \tau] \to P$ of the first player.

The transition rule is specified according to which for any initial position z and initial instant of time $t \in [a, b]$ a new position is obtained

$$z(\tau) = g_t^\tau (z, v(\cdot), u(\cdot)), \quad \forall v(\cdot) \in V(z, t, \tau), \; \forall u(\cdot) \in U(z, t, \tau, v(\cdot)), \quad t < \tau \leqslant b \tag{1.2}$$

Using transition rule (1.2), we construct the operator of programmed absorption in the following manner /4-6/. Let $a \leqslant t < \tau \leqslant b$ and $M \subset Z$, and the point $z \in T_t^\tau (M)$ if an only if for any control $v(\cdot) \in V(z, t, \tau)$ control $u(\cdot) \in U(z, t, \tau, v(\cdot))$ exists for which the point $z(\tau)$ (1.2) belongs to the set M. From this definition it follows that

$$M \subset Y \Rightarrow T_t^\tau (M) \subset T_t^\tau (Y) \tag{1.3}$$

The sufficient conditions for realizing the inclusion (1.1) can be obtained in a number of cases using the following theorem. Let Z be a linear space and $F_i: 2^Z \to 2^Z, i = 1, 2, 3$ and $L_j: 2^Z \to 2^Z, j = 1, 2$ be a multivalued mapping that satisfies the condition

$$M \subset Y \Rightarrow F_i (M) \subset F_i (Y), \quad i = 1, 2 \tag{1.4}$$

Theorem. Let the set X be convex and the mappings F_i and L_j be such that: 1) for any sets M and Y from Z the inclusion $F_i (M + Y) \supset F_i (M) + L_i (Y), \; i = 1, 2$ is satisfied, and 2) a number $0 < \lambda < 1$ exists such that

$$F_1 (L_2 (\lambda X)) \supset \lambda F_3 (X), \; L_1 (F_2 ((1 - \lambda) X)) \supset (1 - \lambda) F_3 (X) \tag{1.5}$$

Then the inclusion

$$F_1 (F_2 (X)) \supset F_3 (X) \tag{1.6}$$

is satisfied.

Proof. From the inclusion (1.4) and the conditions of the theorem we have a chain of inclusions

$$\begin{aligned} F_1 (F_2 (X)) = F_1 (F_2 (\lambda X + (1 - \lambda) X)) \supset F_1 (L_2 (\lambda X)) + \\ F_2 ((1 - \lambda) X) \supset F_1 (L_2 (\lambda X)) + L_1 (F_2 ((1 - \lambda) X)) \supset \\ \lambda F_3 (X) + (1 - \lambda) F_3 (x) \supset F_3 (X) \end{aligned}$$

Let us apply this theorem to some specific classes of games.

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2. Consider a game with the simple motion /4, 7/

$$z^{\cdot} = -f(u, v),\ v \in Q \subset R^l,\ u \in P(v) \subset R^m,\ z \in R^n \tag{2.1}$$

Let the sets $V(z, t, \tau)$ and $U(z, t, \tau, v(\cdot))$ be the totality of constant controls $v\colon [t, \tau] \to Q$ and $u\colon [t, \tau] \to P(v)$. Then writing down the solution of (2.1) for the constant controls, and using the definition of the operator T, we obtain

$$T_t^\tau(M) = \bigcap\nolimits_v (M + (\tau - t) F(v)),\quad v \in Q \tag{2.2}$$
$$F(v) = \{x \in R^n\colon\ x = f(u, v),\ u \in P(v)\}$$

Let the set $V(z, t, \tau)$ be the totality of constant controls $v\colon [t, \tau] \to Q$, and $U(z, t, \tau, v(\cdot))$ the totality of piecewise constant controls $u\colon [t, \tau] \to P(v)$. In this case

$$T_t^\tau(M) = \bigcap\nolimits_v (M + (\tau - t) \operatorname{co} F(v)),\quad v \in Q \tag{2.3}$$

If $V(z, t, \tau)$ and $U(z, t, \tau, v(\cdot))$ are sets of piecewise constant controls $v\colon [t, \tau] \to Q$ and $u(s) \in P(v(s)),\ t \leqslant s \leqslant \tau$, then

$$T_t^\tau(M) = \bigcap\nolimits_{v_i, \lambda_i} \Big(M + (\tau - t) \sum_{i=1}^{k} \lambda_i \operatorname{co} F(v_i)\Big) \tag{2.4}$$

The interaction is taken over all sets $v_1, \ldots, v_k$ from Q, $\lambda_1, \ldots, \lambda_k$, $\lambda_i \geqslant 0$, $\lambda_1 + \ldots + \lambda_k = 1$.

The mappings (2.2)-(2.4) have the following properties:

$$T_t^\tau(M + Y) \supset T_t^\tau(M) + Y,\quad T_t^\tau(\lambda M) = \lambda T_t^b(M) \tag{2.5}$$
$$T_\tau^b((1 - \lambda) M) = (1 - \lambda) T_t^b(M),\quad \lambda = (\tau - t)/(b - t)$$

Using the notation $F_1(M) = T_t^\tau(M)$, $F_2(M) = T_\tau^b(M)$, $F_3(M) = T_t^b(M)$, $L_f(M) = M$, we obtain from property (2.5) the inclusion (1.5). Hence, when X is a convex set, the mappings (2.2)-(2.4) satisfy the inclusion (1.1). From this it follows that when set X is convex, the inclusion

$$\bigcap\nolimits_w (\bigcap\nolimits_v (X + \sigma_2 F(v)) + \sigma_1 F(w)) \supset \bigcap\nolimits_v (X + (\sigma_1 + \sigma_2) F(v)) \tag{2.6}$$

is satisfied. Here w and v are taken from Q, and the numbers $\sigma_1 \geqslant 0, \sigma_2 \geqslant 0$. Let N be a convex set. Let us determine the conditions for the numbers $\sigma_i \geqslant 0$ so that the inclusion

$$\bigcap\nolimits_w (\bigcap\nolimits_v (Y + \sigma_3 N + \sigma_2 F(v)) + \sigma_4 N + \sigma_1 F(w)) \supset \tag{2.7}$$
$$\bigcap\nolimits_v (Y + (\sigma_3 + \sigma_4) N + (\sigma_1 + \sigma_2) F(v))$$

is satisfied. For a convex set Y the inclusion (2.7) is satisfied, if

$$\sigma_3 \sigma_1 \geqslant \sigma_2 \sigma_4 \tag{2.8}$$

Indeed, if $\sigma_4 = 0$, the left side of inclusion (2.7) has the form of the left side of inclusion (2.6) when $X = Y + \sigma_3 N$. If $\sigma_2 = 0$, the inclusion (2.7) becomes an equation. Let $\sigma_2 \sigma_4 > 0$ then, as follows from (2.8), we have $\sigma_1 > 0$. With the notation $\delta = (\sigma_2 \sigma_4)/\sigma_1$ the left side of inclusion (2.7) has the form of the left side of inclusion (2.6) when $X = Y + (\sigma_3 - \delta) N$ and $(\delta/\sigma_2) N + F(v)$ is substituted for $F(v)$. When these expressions are in the right side of inclusion (2.6), we obtain the right side of inclusion (2.7).

Inclusion (2.7) may be used to obtain the sufficient conditions for (1.1) to be satisfied in differential games of the form

$$z^{\cdot} = -\alpha_1(t) u_1 - \alpha_2(t) f(u_2, v),\ v \in Q,\ u_1 \in N,\ u_2 \in P(v) \tag{2.9}$$

where α_1 and α_2 are non-negative continuous functions $z \in R^n$. If, for example, the admissible controls of the players are considered in the class of constants, then

$$T_t^\tau(X) = \bigcap\nolimits_v (X + \langle\alpha_1\rangle_t^\tau N + \langle\alpha_2\rangle_t^\tau F(v)) \tag{2.10}$$

Here and subsequently

$$\langle x\rangle_a^b = \int_a^b x(r)\, dr$$

This shows that the left side of relation (1.1) has the form of the left side of inclusion (2.7) when $\sigma_{1,4} = \langle\alpha_{2,1}\rangle_t^\tau$, $\sigma_{2,3} = \langle\alpha_{2,1}\rangle_\tau^b$. Hence for the image of (2.10) for any convex set $X \subset R^n$ the inclusion (1.1) is satisfied when the condition that corresponds to (2.8) is satisfied.

3. Let us consider a game whose equations of motion have the form

$$z^{\cdot} = v_1 A^c z + \alpha_1(t) v_2 + \alpha_2(t) u,\ z \in R^n,\ v_2 \in Q \subset R^n,\ u \in S \tag{3.1}$$

where A° is a constant skew-symmetric matrix, v_1 is the scalar control restrained by the constraint $|v_1| \leqslant 1$, the functions α_i: $[a, b] \to R$ are continuous and non-negative, the convex compactum Q contains a zero vector, and S is the Euclidean sphere of unit radius in space R^n

Example. The first player has a simple motion in the x, y plane, with its velocity limited in magnitude $(x_1^{\cdot})^2 + (y_1^{\cdot})^2 \leqslant 1$. We introduce the variable φ to denote the direction of motion of the second player, and write its equation of motion in the form $x_2^{\cdot} = v_2 \sin\varphi$, $y_2 = v_2 \cos\varphi$, $\varphi^{\cdot} = v_1$ /8/. Unlike in /8/ we assume that $0 \leqslant v_2 \leqslant v$. If we change to the variables

$$z_1 = (x_1 - x_2)\cos\varphi - (y_1 - y_2)\sin\varphi$$
$$z_2 = (x_1 - x_2)\sin\varphi + (y_1 - y_2)\cos\varphi$$

we obtain an equation of the form (3.1).

Let us write the classes of admissible controls of the players. The number $0 \leqslant \nu \leqslant 1$ is specified. Each admissible control $v(r) = (v_1(r), v_2(r))$ from the set $V(z, t, \tau)$ has the form $|v_1(r)| \leqslant 1$, the control $v_1(r)$ is measurable, when $t \leqslant r \leqslant (1-\nu)t + \nu\tau = t(\nu)$ we have $v_2(r) = 0$, when $v_1(r) = 0$, $v_2(r) \in Q$, the control $v_2(r)$ is measurable when $t(\nu) \leqslant r \leqslant \tau$.

The meaning of such a control in this example is as follows: at the beginning the second player remains in place but alters the direction φ, and then moves along a straight line. The ratio of time taken to alter the direction φ to the time the point moves along the straight line is always constant.

We assume that the set $U(z, t, \tau, v(\cdot))$ of admissible controls of the first player consists of all measurable functions u: $[t, \tau] \to S$. The rule of transition (1.2) for the admissible controls described is obtained from system (3.1) by using the Cauchy formula

$$z(\tau) = e^{A(\tau)}(z(t) + \langle e^{-A}(\alpha_1 v_2 + \alpha_2 u)\rangle_t^{\tau}), \quad A(r) = \langle v_1\rangle_t^r A^\circ \tag{3.2}$$

From the equation $A(r) = A(t(\nu))$, $t(\nu) \leqslant r \leqslant \tau$ and formula (3.2), taking into account the form of the control $v_2(r)$, we have

$$z(\tau) = e^{A(t(\nu))}(z(t) + e^{-A(t(\nu))}\langle\alpha_1 v_2\rangle_{t(\nu)}^{\tau} + \langle e^{-A}\alpha_2 u\rangle_t^{\tau}) \tag{3.3}$$

We further have

$$\{z \in R^n: z = \langle e^{-A}\alpha_2 u\rangle_t^{\tau},\ u: [t, \tau] \to S\} = \langle\alpha_2\rangle_t^{\tau} S \tag{3.4}$$
$$\{z \in R^n: z = \langle\alpha_1 v_2\rangle_{t(\nu)}^{\tau},\ v_2: [t(\nu), \tau] \to Q\} = \langle\alpha_1\rangle_{t(\nu)}^{\tau} Q$$
$$\{B: B = A(t(\nu)),\ v_1: [t, t(\nu)] \to [-1, 1]\} = \{B = yA^\circ: |y| \leqslant \nu(\tau - t)\}.$$

In these formulae the functions u, v_2, v_1 are assumed to be measurable. In the first of equations (3.4) the skew-symmetry of matrix A° is used.

From the definition of the mapping of T and from formulae (3.3) and (3.4) we obtain

$$T_t^{\tau}(M) = \bigcap_y e^{yA^\circ}((M + \langle\alpha_2\rangle_t^{\tau} S) \overset{*}{-} \langle\alpha_1\rangle_{t(\nu)}^{\tau} Q), \quad |y| \leqslant \nu(\tau - t) \tag{3.5}$$

where the symbol $\overset{*}{-}$ denotes the geometric difference of sets /9/. We fix the arbitrary sets $P_i \subset R^n$, and with $\delta \geqslant 0$, $\sigma \geqslant 0$, $\gamma \geqslant 0$ use the notation

$$B(M, \delta, \sigma, \gamma) = \bigcap_y e^{yA^\circ}((M + \sigma P_1) \overset{*}{-} \gamma P_2), \quad |y| \leqslant \delta \tag{3.6}$$

Statement. When $\varepsilon \geqslant 0$ the inclusion

$$B(B(\varepsilon S, \delta_2, \gamma_2, \gamma_2), \delta_1, \gamma_1, \gamma_1) \supset B(\varepsilon S, \delta_1 + \delta_2, \gamma_1 + \gamma_2, \gamma_1 + \gamma_2) \tag{3.7}$$

is satisfied.

Proof. We put

$$F_i(M) = B(M, \delta_i, \gamma_i, \gamma_i), \ L_i(M) = \bigcap_y e^{yA^\circ} M, \ |y| \leqslant \delta_i, \ i = 1, 2 \tag{3.8}$$

From (3.6) it then follows that

$$F_i(M + Y) \supset F_i(M) - L_i(Y)$$

We use the notation $\lambda = \gamma_1/(\gamma_1 + \gamma_2)$ and $F_3(\varepsilon S)$ for the right side of the inclusion (3.7). Then it follows from (3.6) and (3.8) that

$$L_1(F_2((1 - \lambda)\varepsilon S)) = B((1 - \lambda)\varepsilon S, \delta_1 + \delta_2, \gamma_2, \gamma_2) = (1 - \lambda)F_3(\varepsilon S)$$

From the skew-symmetry of the matrix A° there follows the invariance of the set $\beta S, \beta \geqslant 0$ relative to representation L_i. Hence

$$F_1(L_2(\lambda\varepsilon S)) = F_1(\lambda\varepsilon S) \supset B(\lambda\varepsilon S, \delta_1 + \delta_2, \gamma_1, \gamma_1) = \lambda F_3(\varepsilon S)$$

and according to the theorem the inclusion (3.7) is satisfied.

If we now set $P_1 = S$ in (3.6), the equation

$$B(M, \delta, \sigma, \gamma) = B(M + \beta S, \delta, \sigma - \beta, \gamma), \ 0 \leqslant \beta \leqslant \sigma \tag{3.9}$$

will be satisfied.

Let us determine the conditions under which in this case the inclusion

$$B(B(\varepsilon S,\ \delta_2,\ \sigma_2,\ \gamma_2),\ \delta_1,\ \sigma_1,\ \gamma_1) \supset B(\varepsilon S,\ \delta_1+\delta_2,\ \sigma_1+\sigma_2,\ \gamma_1+\gamma_2) \tag{3.10}$$

is satisfied.

Statement 2. When $P_1 = S$, the inclusion (3.10) is satisfied, if

$$\sigma_2\gamma_1 \geqslant \sigma_1\gamma_2 \tag{3.11}$$

Proof. Let $\gamma_1 = 0$. Then the inclusion (3.10) follows from the form of set (3.6).

Let $\gamma_1 > 0$. Then using the notation $\beta = (\sigma_1\gamma_2)/\gamma_1$, we have from (3.9) that $B(\varepsilon S,\ \delta_2,\ \sigma_2,\ \gamma_2) = B((\varepsilon + \sigma_2 - \beta) S, \delta_2, \beta, \gamma_2)$. It follows from this and (3.6) that the left side of inclusion (3.10) is the same as the left side of (3.7), when we substitute in it $\varepsilon + \sigma_2 - \beta$ for ε and $(\beta/\gamma_2) S$ for P_1. With this substitution, taking into account the form of the number β, we find that the right side of inclusion (3.7) is the same as the right side of inclusion (3.10).

From the inequality (3.11) and formula (3.5) we obtain the sufficient conditions for satisfying inclusion (1.1) when $X = \varepsilon S$. This condition has the form

$$\langle\alpha_2\rangle_\tau^b \langle\alpha_1\rangle_{t(\nu)}^{\tau} \geqslant \langle\alpha_2\rangle_t^\tau \langle\alpha_1\rangle_{t(\nu)}^{b},\quad \tau(\nu) = (1-\nu)\tau + \nu b \tag{3.12}$$

This inequality is satisfied, when the functions α_i are constant.

4. Linear differential games with a fixed instant of termination can be reduced by the change of variables /3/ to a game with simple motion.

Let the equations of motion have the form

$$z^{\cdot} = -u + v + f(t),\quad u \in P(t),\quad v \in Q(t),\quad z \in R^n \tag{4.1}$$

The sets $P(t)$ and $Q(t)$ are compacta for each $t \in [a,\ b]$ in R^n, Lebesgue measurable, and in the segment $[a,\ b]$ depend on t. A function $D(t) \geqslant 0$ exists in that segment, such that the sets $P(t)$ and $Q(t)$ are contained in a sphere of radius $D(t)$ whose centre is at the origin of coordinates. The measurable function $f\colon [a,\ b] \to R^n$ satisfies the constraint $|f(t)| \leqslant D(t)$.

Suppose the sets $V(z,\ t,\ \tau)$ and $U(z,\ t,\ \tau,\ v(\cdot))$ are the totality of measurable controls $v\colon [t,\ \tau] \to R^n$ and $u\colon [t,\ \tau] \to R^n$ that satisfy the inclusions $v(r) \in Q(r)$ and $u(r) \in P(r)$. In that case

$$T_t^\tau(M) = R_t^\tau(M) + \langle f\rangle_t^\tau,\ R_t^\tau(M) = (M + \langle P\rangle_t^\tau) \stackrel{*}{-} \langle Q\rangle_t^\tau$$

From this we can obtain that the inclusion (1.1) is satisfied, if

$$R_t^\tau(R_\tau^b(X)) \supset R_t^b(X) \tag{4.2}$$

Statement 3. Let convex sets P_i and Q_i and a number $0 < \lambda < 1$ exist such that

$$\langle P\rangle_\tau^b = P_1 + P_2,\quad \langle Q\rangle_t^\tau = Q_1 + Q_2 \tag{4.3}$$

$$(1-\lambda)\langle P\rangle_t^\tau = \lambda P_1,\ \lambda\langle Q\rangle_\tau^b = (1-\lambda) Q_1 \tag{4.4}$$

Then the inclusion (4.2) is satisfied for a convex set X.

Proof. We use the notation $X_1 = X + P_2$. Then from (4.2) and (4.3), and the properties of the geometrical difference $A \stackrel{*}{-} (B + C) = (A \stackrel{*}{-} B) \stackrel{*}{-} C$ we obtain

$$R_t^\tau(R_\tau^b(X)) = N \stackrel{*}{-} Q_2,\quad N = ((X_1 + P_1) \stackrel{*}{-} \langle Q\rangle_\tau^b + \langle P\rangle_t^\tau) \stackrel{*}{-} Q_1 \tag{4.5}$$

Taking into account the convexity of the set X_1, we conclude that

$$N \supset ((1-\lambda) X_1 + P_1) \stackrel{*}{-} \langle Q\rangle_\tau^b + (\lambda X_1 + \langle P\rangle_t^\tau) \stackrel{*}{-} Q_1 \tag{4.6}$$

From (4.4) it follows that

$$\langle P\rangle_t^\tau \supset \lambda(\langle P\rangle_t^\tau + P_1),\ P_1 \supset (1-\lambda)\ (\langle P\rangle_t^\tau + P_1)$$
$$\lambda(\langle Q\rangle_\tau^b + Q_1) \supset Q_1,\ \langle Q\rangle_\tau^b \subset (1-\lambda)(\langle Q\rangle_\tau^b + Q_1)$$

From here and (4.6) we have the inclusion

$$N \supset (X_1 + \langle P\rangle_t^\tau + P_1) \stackrel{*}{-} (\langle Q\rangle_\tau^b + Q_1)$$

From this, taking into account $X_1 = X + P_2$ and the relations (4.3) we obtain that the right sides of the first of equations (4.5) contains the set

$$(X + P_2 + \langle P\rangle_t^\tau + P_1) \stackrel{*}{-} (\langle Q\rangle_\tau^b + Q_1 + Q_2) = T_t^b(X)$$

Consider the case when for all $t \leqslant \tau \leqslant b$

$$\langle P\rangle_t^\tau = A(\langle\alpha_1\rangle_t^\tau, \ldots, \langle\alpha_k\rangle_t^\tau),\ \langle Q\rangle_t^\tau = B(\langle\beta_1\rangle_t^\tau, \ldots, \langle\beta_m\rangle_t^\tau) \tag{4.7}$$

where $\alpha_i(r)$ and $\beta_i(r)$ are continuous non-negative functions, and the sets $A(\alpha)$ and $B(\beta)$ for each array of non-negative sets of the numbers $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_m)$ are convex and have the following linear properties:

$$A(\alpha^1+\alpha^2) = A(\alpha^1) + A(\alpha^2), \lambda A(\alpha) = A(\lambda\alpha), \lambda \geqslant 0$$
$$B(\beta^1+\beta^2) = B(\beta^1) + B(\beta^2), \lambda B(\beta) = B(\lambda\beta), \lambda \geqslant 0 \tag{4.8}$$

From formulae (4.7) and (4.8) it follows that conditions (3.4) and (4.4) will be satisfied, if there are numbers $0<\lambda<1, \alpha_i \geqslant 0, \beta_i \geqslant 0$ such that

$$(1-\lambda)\langle\alpha_i\rangle_t^\tau = \lambda\alpha_i,\ \alpha_i \leqslant \langle\alpha_i\rangle_\tau^b,\ i=1,\ldots,k$$
$$\lambda\langle\beta_j\rangle_\tau^b = (1-\lambda)\beta_j,\ \beta_j \leqslant \langle\beta_j\rangle_t^\tau,\ j=1,\ldots,m$$

These conditions will be satisfied, if for all $i=1,\ldots,k$ and $j=1,\ldots,m$ we have

$$\langle\alpha_i\rangle_t^\tau \leqslant \lambda\langle\alpha_i\rangle_t^b,\ \lambda\langle\beta_j\rangle_t^b \leqslant \langle\beta_j\rangle_t^\tau,\ 0<\lambda<1$$

From here we obtain the condition which when satisfied results in the satisfaction of (4.3) and (4.4)

$$\max_i(\langle\alpha_i\rangle_t^\tau/\langle\alpha_i\rangle_t^b) \leqslant \min_j(\langle\beta_j\rangle_t^\tau/\langle\beta_j\rangle_t^b) \tag{4.9}$$

We shall now give some examples of multivalued functions that satisfy (4.8). If $A_1,\ldots,A_k$ are convex compacts in R^n, then $A(\alpha) = \alpha_1A_1+\ldots+\alpha_kA_k$ satisfies the condition (4.8).

Let $A_i, i=1,\ldots,n+1$ be defined by the scalar product of inequalities $(x_i,x)\leqslant 1$ in R^n. Here $x_1,\ldots,x_n,x_{n+1}$ are vectors from R^n and the first of them are linearly independent, and the coefficients f_i in expansion $x_{n+1} = f_1x_1+\ldots+f_nx_n$ are negative.

Consider the set

$$A(\alpha_1,\ldots,\alpha_{n+1}) = \cap(\alpha_iA_i) = \{x\in R^n : (x_i,x)\leqslant\alpha_i, i=1,\ldots,n+1\} \tag{4.10}$$

in which α_i are non-negative. Then, as shown in /10/, the set (4.10) satisfies condition (4.8).

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SINGULAR PERTURBATIONS IN A CLASS OF PROBLEMS OF OPTIMAL CONTROL WITH INTEGRAL CONVEX CRITERION*

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The problem of optimal control is investigated with a linear law of motion and convex quality criterion. A small positive parameter appears in front of the derivatives of some of the unknowns in the law of motion. The behaviour of the optimal solution is studied when the small parameter approaches zero with some assumptions that are different from thos encountered in the literature.

1. Controlled objects whose law of motion is

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